CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems to usual applications.

G.H.E. Duchamp

Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
Karol A. Penson, Darij Grinberg, Hoang Ngoc Minh, C. Lavault,
C. Tollu, N. Behr, V. Dinh, C. Bui,
Q.H. Ngô, N. Gargava, S. Goodenough.

CIP seminar,

Friday conversations:

For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

Disclaimer. – The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

CCRT[19] Functional and Topological Questions I.

1. Last time, we stopped at three items
   1. Iterated integrals
   2. NCDE $S' = MS$ with asymptotic condition
   3. The topology of $\mathcal{H}(\Omega)$
2. ... and stated some open problems relative to the tree of holomorphic functions generated (continuity, Baire classes)
3. Today we will explore a domain theory dedicated to the Li_w for $w \in X^*x_1$.
4. Some concluding remarks.

Introduction

Goal of this talk. – In

Three variations on the linear independence of grouplikes in a coalgebra.

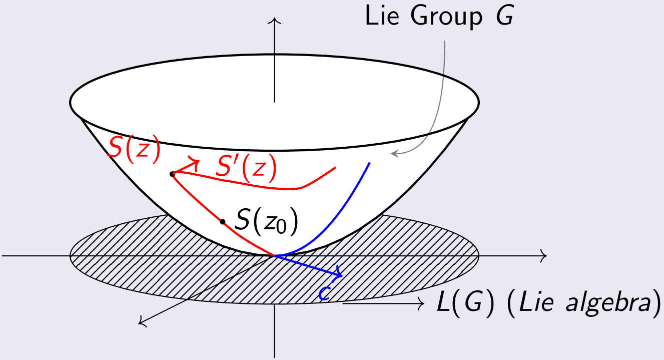
(GD, Darij Grinberg, Hoang Ngoc Minh, see [15]), we have a very general theorem. It generalises a result re-appearing in different forms (since the seminal work of Moss E. Sweedler) and which roughly says "the elements of the group of characters of a bialgebra are linearly independant" (see [1, 41, 43]). Our generalization consists in enlarging the set of scalars to a sort of "polynomial nilpotent convolution algebra" (the nilpotence is controlled there by an increasing filtration). Applied to the bialgebra $(\mathbb{C}\langle X\rangle, \mathtt{conc}, 1_{X^*}, \Delta_{shuffle}, \epsilon)$, this theorem gives the following result:

The stars of the type $(\sum_{x\in X} \alpha_x\, x)^*$ *are linearly independent with respect to* $\mathbb{C}\langle X\rangle$ *within* $\mathbb{C}\langle\langle X\rangle\rangle$.

This is an algebraic result.
The goal of this talk is therefore a small route between algebra and analysis i.e. a case study (through the shuffle character Li) about the transformation of this group and "to which extent" analysis (and the host of limiting processes it provides) makes the "big picture" more accurate.

Solutions as paths drawn on the Magnus group.

1. The paradigm we will use in the future is that, if $S(z)$ (each coordinate holomorphic), drawn on the Magnus group is such that
   1. $S(z_0)$ belongs to some closed subgroup G
   2. $\mathbf{d}(S)S^{-1}[z] = M(z)$ belongs, for all $z \in \Omega$ to the tangent space $T_1(G)$.
   3. Here $S(z_0)$ is replaced by a limit condition (as if $z_0 \in \overline{\Omega}$) we will exploit the subgroup (i.e. Hausdorff) algebraically.



Starting point: the ladder of CCRT[18].

2. Starting point $(\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*})$

$$
\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
\mathbb{C}\langle X\rangle \shuffle \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \shuffle \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \xrightarrow{\mathrm{Li}_\bullet^{(2)}} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\uparrow & \nearrow & \\
\mathbb{C}\langle X\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & &
\end{array}
$$

3. These extensions, as well as: topological, functional and closed subgroup properties will be the subject of forthcoming talks.

Explicit construction of Li

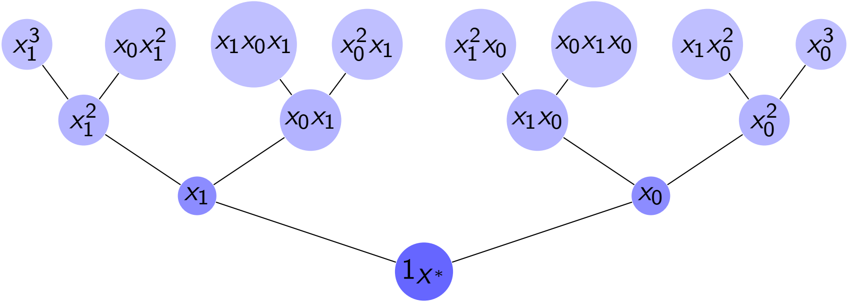
Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

$$\alpha_0^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_0^z \alpha_0^s(u) \frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_1^z \alpha_0^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \\ \int_0^z \alpha_0^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0 \ . \end{cases} \tag{1}$$

Of course, the third line of this recursion implies

$$\alpha_0^z(x_0^n) = \frac{log(z)^n}{n!}$$

one can check that (a) all the integrals (although improper for the fourth line) are well defined (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)\, w$ satisfies (2). We then have $\alpha_0^z = \mathrm{Li}$.



As an example, we compute some coefficients

$$\langle \mathrm{Li} \,|x_0^n\rangle = \frac{log(z)^n}{n!} \quad ; \quad \langle \mathrm{Li} \,|x_1^n\rangle = \frac{(-log(1-z))^n}{n!}$$

$$\langle \mathrm{Li} \,|x_0x_1\rangle = \mathrm{Li}_2(z) = \sum_{n\geq 1}\frac{z^n}{n^2} \quad ; \quad \langle \mathrm{Li} \,|x_1x_0\rangle = \langle \mathrm{Li} \,|x_1 \shuffle x_0 - x_0x_1\rangle(z)$$

$$\langle \mathrm{Li} \,|x_0^2x_1\rangle = \mathrm{Li}_3(z) = \sum_{n\geq 1}\frac{z^n}{n^3} \quad ; \quad \langle \mathrm{Li} \,|x_1x_0\rangle = (-log(1-z))log(z) - \mathrm{Li}_2(z)$$

$$\langle \mathrm{Li} \,|x_0^{r-1}x_1\rangle = \mathrm{Li}_r(z) = \sum_{n\geq 1}\frac{z^n}{n^r} \quad ; \quad \langle \mathrm{Li} \,|x_1^2x_0\rangle = \langle \mathrm{Li} \,|\frac{1}{2}(x_1 \shuffle x_1 \shuffle x_0) - (x_1 \shuffle x_0x_1) + x_0x_1^2\rangle$$

Li From Noncommutative Diff. Eq.

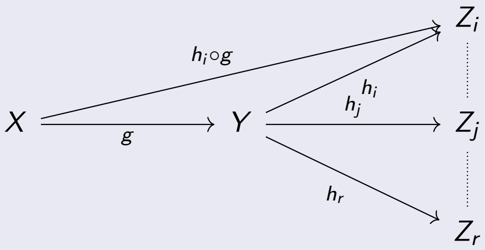
The generating series $S = \sum_{w \in X^*} Li(w)$ satisfies (and is unique to do so)

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \lim_{\substack{z \to 0 \\ z \in \Omega}} S(z) e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} \end{cases} \quad (2)$$

with $X = \{x_0, x_1\}$. This is, up to the sign of x_1, the solution G_0 of Drinfel'd [12] for KZ3. We define this unique solution as Li. All Li_w are $\mathbb{C}$- and even $\mathbb{C}(z)$-linearly independant (see CAP 17's talk "*Linear independance without monodromy*").

The category **Top**, initial topologies.

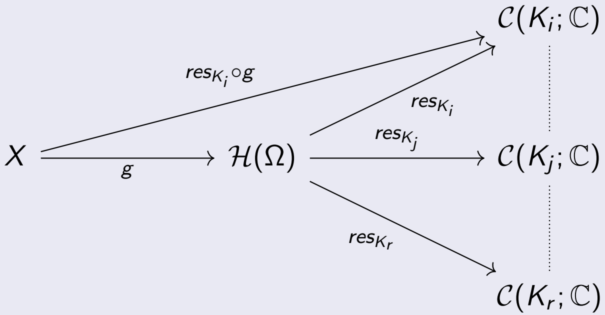
4. We now use a very very general construction, well suited both for series and holomorphic functions (and many other situations), that of initial topologies (see [40] and, for a detailed construction [5], Ch1 §2.3). Let Y be a set together with a family of maps $h_i : \ Y \to Z_i$ where Z_i are all topological spaces.



5. It exists a unique topology τ on Y such that, for all $X \in$**Top**

$$g \text{ is continuous } \iff (\forall i \in I)(h_i \circ g \text{ is continuous })$$

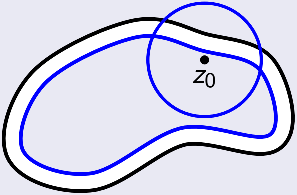
4. Now, the topology on $\mathcal{H}(\Omega)$ is defined by the family of maps $res_K : \mathcal{H}(\Omega) \to \mathcal{C}(K;\mathbb{C})$



5. So $\mathcal{H}(\Omega)$ is a locally convex TVS whose topology is defined by the family of seminorms $(|| \; ||_K)_{K \in \mathfrak{K}(\Omega)}$ (where $(||f||_K) = sum_{z \in K}|f(z)|)$.

Topology of $\mathcal{H}(\Omega)$ cont'd.

4. In fact, every $\Omega \subset \mathbb{C}$ is σ-compact, this means that one can construct a sequence $(K_n)_{n\geq 1}$ of compacts i.e. $(\forall K \in \mathfrak{K}(\Omega))(\exists n \geq 1)(K \subset K_n)$ therefore $\mathcal{H}(\Omega)$ is a complete (hence closed) subset of the product $\Pi_{n\geq 1}\mathcal{C}(K_n; \mathbb{C})$ (for the topology on the cube, see a next CCRT).
$$K_n = \{z \in \Omega \mid d(z, z_0) \leq n \text{ and } d(z, \mathbb{C} \setminus \Omega) \geq \frac{1}{n}\}.$$



5. We will see more (step-by-step and starting from scratch) on the topology of the cube and separability in the CCRT devoted to convergence questions).

Domain of Li (definition)

In order to extend indexation of Li to series, we define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S=\sum_{n\geq 0} S_n$ (decomposition by homogeneous components) such that $\sum_{n\geq 0} Li_{S_n}(z)$ converges unconditionally for compact convergence in Ω. One sets

$$Li_S(z) := \sum_{n\geq 0} Li_{S_n}(z) \tag{3}$$

Starting the ladder

$$\begin{array}{ccc} (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\ \downarrow & & \downarrow \\ (\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \end{array}$$

Examples

$$Li_{x_0^*}(z)=z,\quad Li_{x_1^*}(z)=(1-z)^{-1},\ Li_{\alpha x_0^*+\beta x_1^*}(z)=z^{\alpha}(1-z)^{-\beta}$$

Properties of the extended Li

Proposition

With this definition, we have

1. $Dom(Li)$ is a shuffle (unital) subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ and so is
$$Dom^{rat}(Li) := Dom(Li) \cap \mathbb{C}^{rat}\langle\langle X\rangle\rangle$$
2. For $S, T \in Dom(Li)$, we have still $\mathrm{Li}_{1_{X^*}} = 1_\Omega$ and
$$\mathrm{Li}_{S \,\text{ш}\, T} = \mathrm{Li}_S \,.\, \mathrm{Li}_T$$

Examples and counterexamples

For $|t| < 1$, one has $(tx_0)^* x_1 \in Dom(Li, D)$ (D being the open unit slit disc and $Dom(Li, D)$ defined similarly), whereas $x_0^* x_1 \notin Dom(Li, D)$. Indeed, we have to examine the convergence of $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z)$, but, for $z \in]0, 1[$, one has $0 < z < \mathrm{Li}_{x_0^n x_1}(z) \in \mathbb{R}$ and therefore, for these values $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z) = +\infty$. Furthermore one can show that, for $|t| < 1$,
$$\mathrm{Li}_{(tx_0)^* x_1}(z) = \sum_{n\geq 1} \frac{z^n}{n-t}$$

Passing to harmonic sums $\mathrm{H}_w,\ w \in Y^*$

Polylogarithms having a removable singularity at zero

The following proposition helps us characterize their indices.

Proposition

Let $f(z) = \langle \mathrm{Li} \,|P\rangle = \sum_{w \in X^*} \langle P|w\rangle \,\mathrm{Li}_w$. *The following conditions are equivalent*

i) *f can be analytically extended around zero*

ii) $P \in \mathbb{C}\langle X\rangle x_1 \oplus \mathbb{C}.1_{X^*}$

We recall the expansion (for $w \in X^* x_1 \sqcup \{1_{X^*}\},\ |z| < 1$)

$$\frac{\mathrm{Li}_w(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{\pi_Y(w)}(N)\, z^N \tag{4}$$

Global and local domains

This proposition and the lemma lead us to the following definitions.

1. *Global domains*.–
   Let $\emptyset \neq \Omega \subset \widetilde{B}$ (with $B = \mathbb{C} \setminus \{0,1\}$), we define $Dom_\Omega(Li) \subset \mathbb{C}\langle\langle X \rangle\rangle$ to be the set of series $S = \sum_{n \geq 0} S_n$ (with $S_n = \sum_{|w|=n} \langle S|w \rangle\, w$ each homogeneous component) such that $\sum_{n\in\mathbb{N}} Li_{S_n}$ is unconditionally convergent for the compact convergence (UCC) [31].
   As examples, we have Ω_1, the doubly cleft plane then $Dom(\mathrm{Li}) := Dom_{\Omega_1}(\mathrm{Li})$ or $\Omega_2 = \widetilde{B}$
2. *Local domains around zero (fit with H-theory)*.–
   Here, we consider series $S \in (\mathbb{C}\langle\langle X \rangle\rangle x_1 \oplus \mathbb{C}\, 1_{X^*})$ (i.e. $supp(S) \cap Xx_0 = \emptyset$).
   We consider radii $0 < R \leq 1$, the corresponding open discs $D_R = \{z \in \mathbb{C} |\ |z| < R\}$ and define

$$\begin{aligned} Dom_R(\mathrm{Li}) &:= \{S = \Sigma_{n\geq 0} S_n \in (\mathbb{C}\langle\langle X \rangle\rangle x_1 \oplus \mathbb{C}1_\Omega) | \sum_{n\in\mathbb{N}} Li_{S_n} \text{ (UCC) in } D_R\} \\ Dom_{loc}(\mathrm{Li}) &:= \cup_{0<R\leq 1} Dom_R(\mathrm{Li}). \end{aligned}$$

Properties of the domains

Theorem A

1. For all $\emptyset \neq \Omega \subset \widetilde{B}$, $Dom_{\Omega}(\mathrm{Li})$ is a shuffle subalgebra of $\mathbb{C}\langle\langle X \rangle\rangle$ and so are the $Dom_R(\mathrm{Li})$.
2. $R \mapsto Dom_R(\mathrm{Li})$ is strictly decreasing for $R \in]0,1]$.
3. All $Dom_R(\mathrm{Li})$ and $Dom_{loc}(\mathrm{Li})$ are shuffle subalgebras of $\mathbb{C}\langle\langle X \rangle\rangle$ and $\pi_Y(Dom_{loc}(\mathrm{Li}))$ is a stuffle subalgebra of $\mathbb{C}\langle\langle Y \rangle\rangle$.
4. Let $T(z) = \sum_{N \geq 0} a_N z^N$ be a Taylor series i.e. such that $\limsup_{N \to +\infty} |a_N|^{1/N} = B < +\infty$, then the series

$$S = \sum_{N \geq 0} a_N (-(-x_1)^+)^{ш N} \tag{5}$$

is summable in $\mathbb{C}\langle\langle X \rangle\rangle$ (with sum in $\mathbb{C}\langle\langle x_1 \rangle\rangle$) and $S \in Dom_R(Li)$ with $R = \frac{1}{B+1}$ and $\mathrm{Li}_S = T(z)$.

Theorem A/2

5. Let $S \in Dom_R(\mathrm{Li})$ and $S = \sum_{n\geq 0} S_n$ (homogeneous decomposition), we define[a] $N \mapsto \mathrm{H}_{\pi_Y(S)}(N)$ by

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(S)}(N) z^N \; . \tag{6}$$

Moreover, for all $r \in]0, R[$, we have

$$\sum_{n,N\geq 0} |\mathrm{H}_{\pi_Y(S_n)} r^N| < +\infty, \tag{7}$$

in particular, for all $N \in \mathbb{N}$ the series (of complex numbers) $\sum_{n\geq 0} \mathrm{H}_{\pi_Y(S_n)}(N)$ converges absolutely to $\mathrm{H}_{\pi_Y(S)}(N)$.

[a] This definition is compatible with the old one when S is a polynomial.

Theorem A/3

6. Conversely, let $Q \in \mathbb{C}\langle\langle Y \rangle\rangle$ with $Q = \sum_{n\geq 0} Q_n$ (decomposition by weights), we suppose that it exists $r \in]0, 1]$ such that

$$\sum_{n,N\geq 0} |\mathrm{H}_{Q_n}(N) r^N| < +\infty \tag{8}$$

in particular, for all $N \in \mathbb{N}$, $\sum_{n\geq 0} \mathrm{H}_{Q_n}(N) = \ell(N) \in \mathbb{C}$ unconditionally.
Under such circumstances, $\pi_X(Q) \in Dom_r(\mathrm{Li})$ and, for all $|z| \leq r$

$$\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\geq 0} \ell(N) z^N, \tag{9}$$

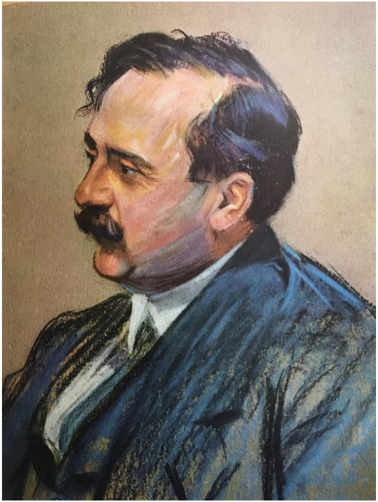
 

Figure: Jacques Hadamard and Paul Montel.

Continuing the ladder

$$
\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
\mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \xrightarrow{\mathrm{Li}_\bullet^{(2)}} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\uparrow & \nearrow & \\
\mathbb{C}\langle X\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & &
\end{array}
$$

We have, after a theorem by Leopold Kronecker,

$$
\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \left\{\frac{P}{Q}\right\}_{\substack{P,Q\in\mathbb{C}[x]\\ Q(0)\neq 0}} \tag{10}
$$

On the right: freeness without monodromy

Theorem (Deneufchâtel, GHED,Minh & Solomon, 2011 [?])

Let $(\mathcal{A}, \partial)$ be a k-commutative associative differential algebra with unit and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $\partial(\mathcal{C}) \subset \mathcal{C}$). We suppose that $k = ker(\partial)$ and that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS \; ; \; \langle S|1\rangle = 1 \text{ with } M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X \rangle\rangle \tag{11}$$

(i.e. M is a homogeneous series of degree 1)
The following conditions are equivalent :

1. *The family $(\langle S|w\rangle)_{w \in X^*}$ of coefficients of S is (linearly) free over $\mathcal{C}$.*
2. *The family of coefficients $(\langle S|x\rangle)_{x \in X \cup \{1_{X^*}\}}$ is (linearly) free over $\mathcal{C}$.*
3. *The family $(u_x)_{x \in X}$ is such that, for $f \in \mathcal{C}$ et $\alpha_x \in k$*

$$\partial(f) = \sum_{x \in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0).$$

A useful property

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Independence of characters with respect to polynomials

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I came across the following property :

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra and ϵ, its counit, is the only character of $\mathcal{U} \to k$ which vanishes on $\mathfrak{g}$.

Set $\mathcal{U}_+ = \ker(\epsilon)$. We build the following filtrations ($N \geq 0$)

$$\mathcal{U}_N = \mathcal{U}_+^N = \underbrace{\mathcal{U}_+ \dots \mathcal{U}_+}_{N \text{ times}} \qquad (1)$$

(in fact $\mathcal{U}_0 = \mathcal{U}, \mathcal{U}_{N+1} = \mathcal{U}.\mathcal{U}_N$) and, for $N \geq -1$

$$\mathcal{U}_N^* = \mathcal{U}_{N+1}^{\perp} = \{f \in \mathcal{U}^* | (\forall u \in \mathcal{U}_{N+1})(f(u) = 0)\} \qquad (2)$$

the first one is decreasing and the second one increasing (in particular $\mathcal{U}_{-1}^* = \{0\},\ \mathcal{U}_0^* = k.\epsilon$).

One shows easily that, for $p, q \geq 0$ (with $\diamond$ as the convolution product)

$$\mathcal{U}_p^* \diamond \mathcal{U}_q^* \subset \mathcal{U}_{p+q}^*$$

so that $\mathcal{U}_\infty^* = \cup_{n \geq 0} \mathcal{U}_n^*$ is a convolution subalgebra of $\mathcal{U}^*$.

Now, we can state the

> **Theorem A** : The set of characters of $(\mathcal{U}, ., 1_{\mathcal{U}})$ is linearly free w.r.t. $\mathcal{U}_\infty^*$.

Remarks i) $\mathcal{U}_\infty^*$ is a commutative k-algebra.

ii) The title comes from the fact that, with $(k\langle X \rangle, \text{conc}, 1)$ (non commutative polynomials), k a

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8 characters on a finite group with `extremal' behaviour

1 Primitive Characters

Left and then right: the arrow $\mathrm{Li}_\bullet^{(1)}$

Proposition

i. The family $\{x_0^*, x_1^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$ within $(\mathbb{C}\langle\langle X\rangle\rangle^{\mathrm{rat}}, ш, 1_{X^*})$.

ii. $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ is a free module over $\mathbb{C}\langle X\rangle$, the family $\{(x_0^*)^{ш\, k} ш\, (x_1^*)^{ш\, l}\}_{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ is a $\mathbb{C}\langle X\rangle$-basis of it.

iii. As a consequence, $\{w\, ш\, (x_0^*)^{ш\, k} ш\, (x_1^*)^{ш\, l}\}_{\substack{w\in X^* \\ (k,l)\in\mathbb{Z}\times\mathbb{N}}}$ is a $\mathbb{C}$-basis of it.

iv. $\mathrm{Li}_\bullet^{(1)}$ is the unique morphism from $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*]$ to $\mathcal{H}(\Omega)$ such that

$$x_0^* \to z,\ (-x_0)^* \to z^{-1} \text{ and } x_1^* \to (1-z)^{-1}$$

v. $\mathrm{Im}(\mathrm{Li}_\bullet^{(1)}) = \mathcal{C}_\mathbb{Z}\{\mathrm{Li}_w\}_{w\in X^*}$.

vi. $\ker(\mathrm{Li}_\bullet^{(1)})$ is the (shuffle) ideal generated by $x_0^*\, ш\, x_1^* - x_1^* + 1_{X^*}$.

Sketch of the proof (pictorial)

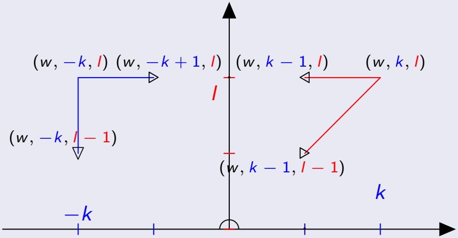

Figure: Rewriting $\mathcal{J}$-**Mod** of $\{w ш (x_0^*)^{ш\, k} ш (x_1^*)^{ш\, l}\}_{k\in\mathbb{Z}, l\in\mathbb{N}, w\in X^*}$.

Open problems and some solved (recall)

6. Do we have $\mathcal{H}(\Omega) = \overline{Im(Dom(\mathrm{Li}))}$ $(= \overline{Im(\mathrm{Li})})$? (in other words does it exist inaccessible $f \in \mathcal{H}(\Omega)$?)
7. If $z_0 \notin \Omega$, does $1/(z - z_0)$ belong to $Im(\mathrm{Li})$? ($z_0 \in \overline{\Omega}$ and $z_0 \notin \overline{\Omega}$)
8. (Solved) Are there non-rational series in $Dom(\mathrm{Li})$? (answer **yes**)
9. (Solved) Is $\mathbb{C}^{rat}\langle\langle X \rangle\rangle$ contained in $Dom(\mathrm{Li})$ (answer **no**)
10. What is the topological complexity of $Dom(\mathrm{Li})$ in the **Borel hierarchy** (Addison notations, see [23] for details and use the convenient framework of polish spaces [6], ch IX).
11. **Borel hierarchy**: We recall that this hierarchy is indexed by ordinals and defined as follows
    1. A set is in $\mathbf{\Sigma}_1^0$ if and only if it is open.
    2. A set is in $\mathbf{\Pi}_\alpha^0$ if and only if its complement is in $\mathbf{\Sigma}_\alpha^0$.
    3. A set A is in $\mathbf{\Sigma}_\alpha^0$ for $\alpha > 1$ if and only if there is a sequence of sets $A_1, A_2, \ldots$ such that each A_i is in $\mathbf{\Pi}_{\alpha_i}^0$ for some $\alpha_i < \alpha$ and $A = \bigcup A_i$.
    4. A set is in $\mathbf{\Delta}_\alpha^0$ if and only if it is both in $\mathbf{\Sigma}_\alpha^0$ and in $\mathbf{\Pi}_\alpha^0$.

Open problems and some solved (recall)/2

12. From slide (8), one can remark that the iterated integrals are based on two integrators, informally defined as

$$\iota_1(f) := \int_0^z f(s)\frac{ds}{1-s} \ ; \ \iota_0(f) := \int_{z_0}^z f(s)\frac{ds}{s} \text{ with } z_0 \in \{0,1\} \qquad (12)$$

ι_1 is defined and continous on $\mathcal{H}(\Omega)$ and ι_0 is defined on $span_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}$ [a] (context-dependent) and not continuous [17] on this set (see below).
Problem What is the Baire class of ι_0 ?

13. Recall that $\mathfrak{K}(\Omega)$ admits a cofinal sequence $(K_n)_{n\in\mathbb{N}}$ of compacts i.e. $(\forall K \in \mathfrak{K}(\Omega))(\exists n \in \mathbb{N})(K \subset K_n)$ therefore $\mathcal{H}(\Omega)$ is a complete (hence closed) subset of the product $\Pi_{n\in\mathbb{N}}\mathcal{C}(K_n;\mathbb{C})$.

14. An alternative way is to define K_n as the family of closed disks contained in Ω such that
    - the center has rational coordinates
    - the radius in rational

[a] It can be a little bit extended, see our paper [17].

Li as a shuffle character (Lie theoretical proof, sketched).

15. Recall what has been said in one of our previous CCRT about the Hausdorff group of the Hopf algebra $(\mathbb{C}\langle X\rangle, ш, 1_{X^*}, \Delta_{\texttt{conc}}, \epsilon)$ (the antipode exists but is not needed here). Let us recall its features
    1. The shuffle product between two words is defined by recursion or duality (see our paper [14])
    2. $\Delta_{\texttt{conc}}$, the dual of conc is defined, within $\mathbb{C}\langle X\rangle$, by duality
    $$\langle\Delta_{\texttt{conc}}(w)|u\otimes v\rangle = \langle w|uv\rangle$$
    or combinatorially $\Delta_{\texttt{conc}}(w) = \sum_{uv=w} u\otimes v$
    3. $\epsilon(P) = \langle P|1_{X^*}\rangle$
16. For every Hopf algebra $(\mathcal{B}, \mu, 1_{\mathcal{B}}, \Delta, \epsilon)$, the set $\Xi(\mathcal{B})$ of characters of $(\mathcal{B}, \mu, 1_{\mathcal{B}})$ is a group under convolution (a monoid in case of a general bialgebra, see our paper [15] Prop. 5.6).
17. Here, due to the fact that $\mathbb{C}$ is a field, we can characterize the group of shuffle characters $\Xi(\mathcal{B})$ by the (algebraic) equations

$$\langle S|1_{X^*}\rangle = 1_{\mathbb{C}} \ ; \ \Delta_{ш}(S) = S\otimes S \qquad (13)$$

Li as a shuffle character/2

18 Let us now consider an evolution equation $S' = M.S$ in $\mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ with a primitive multiplier i.e., for all $z \in \Omega$,

$$\Delta_{ш}(M(z)) = M(z) \otimes 1_{X^*} + 1_{X^*} \otimes M(z)$$

19 Then, if S is group-like (for $\Delta_{ш}$) at one point $z_0 \in \Omega$, it is group-like everywhere (we will see that the point can be remote, or frontier).

20 Let us have a look at the proof, from which we will deduce the version with asymptotic initial condition. We propose the first following statement

Proposition

Let be given, within $\mathcal{H}(\Omega)\langle\langle X\rangle\rangle$, the following evolution equation

$$S' = M.S \ ; \ S(z_0) = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \tag{14}$$

we suppose that, for all $z \in \Omega$, $M(z)$ is primitive (for $\Delta_{ш}$).

Then, *for all $z \in \Omega$, $S(z)$ is group-like (for $\Delta_{ш}$). This means that S is a character of $(\mathcal{H}(\Omega)\langle X\rangle, ш, 1_{X^*})$.*

Li as a shuffle character/3

Proof

21. Firstly, we transform (14) by $\Delta_{ш}$ (which commute - easy exercise - with derivation)

$$\Delta_{ш}(S)' = \Delta_{ш}(S') = \Delta_{ш}(M).\Delta_{ш}(S)\ ;\ \Delta_{ш}(S(z_0)) = 1 \otimes 1$$

22. Taking into account that M is primitive, we get

$$\Delta_{ш}(S)' = (M \otimes 1 + 1 \otimes M).\Delta_{ш}(S)\ ;\ \Delta_{ш}(S(z_0)) = 1 \otimes 1 \qquad (15)$$

23. Let us see what happens to $S \otimes S$

$$(S \otimes S)' \stackrel{(1)}{=} S' \otimes S + S \otimes S' = MS \otimes S + S \otimes MS = (M \otimes 1 + 1 \otimes M).(S \otimes S) \qquad (16)$$

24. We see that $\Delta_{ш}(S)$ and $S \otimes S$ satisfy the same evolution equation (same multiplier) and same initial condition (at z_0).

Li as a shuffle character/4

Proof

25. Then, for every $z \in \Omega$, we have $\Delta_{ш}(S(z)) = S(z) \otimes S(z)$ (and still $\langle S(z)|1_{X^*}\rangle = 1_{\mathbb{C}}$).
26. Finally, as $S(z)$ is a character for every $z \in \Omega$, we get that S is a character of $(\mathcal{H}(\Omega)\langle X\rangle, ш, 1_{X^*})$.

Let us try this one.

27. As an excellent exploratory exercise, we can try the multiplier
$$u_0.x_0 + u_1.x_1 + u_2.[x_0, x_1]$$
with $u_i \in \mathcal{H}(\Omega)$.
28. For example, with
$u_0 = 1/z,\ u_1 = 1/(1-z),\ u_2 = \big(2\,\mathrm{Li}_2 + \log(z)\log(1-z)\big)'$ we do not have linear independence of $(\langle S|w\rangle)_{w\in X^*}$.

What is the condition ? (Forthcoming talk)

Some shuffle subalegbras of $Im(\mathrm{Li}_\bullet)$ and their images.

29. Starting point $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$

$$
\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \overset{\mathrm{Li}_\bullet}{\hookrightarrow\!\!\!\twoheadrightarrow} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\big\downarrow & & \big\downarrow \\
(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \overset{\mathrm{Li}_\bullet^{(1)}}{\twoheadrightarrow} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}\langle X\rangle \, ш \, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \, ш \, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \overset{\mathrm{Li}_\bullet^{(2)}}{\longrightarrow} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\big\uparrow\!\!\uparrow & \nearrow & \\
\mathbb{C}\langle X\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & &
\end{array}
$$

30. These extensions, as well as closed subgroup properties will be the subject of forthcoming talks.

Concluding remarks

1. We have started with iterated integrals (trees, $S' = MS$, primitives, sectioned subalgebras towards integro-differential rings.)
2. Generating series of iterated integrals satisfy a very special class of NCDE $S' = MS$ (i.e. with multiplier of the type $M = \sum_{x \in X} u_x\, x$ and initial condition $S(z_0) = 1$).
3. This entails that the solution of (NCDE + Init) is a shuffle character.
4. Other solutions with then same multiplier share this property (shuffle character), i.e. the solutions with asymptotic initial condition.
5. In particular the arrow Li $(Dom(\mathrm{Li}))$
6. Integrators ι_i (discontinuity of ι_0, last time)
7. Open questions
    1. Topological complexity of $Dom(\mathrm{Li}), \mathrm{Li}(Dom(\mathrm{Li}))$
    2. Closure of $\mathrm{Li}(Dom(\mathrm{Li}))$
    3. Baire class of ι_0

Concluding remarks/2

8. Extending the domain of polylogarithms to (some) rational series permits the projection of rational identities. Such as

$$(\alpha x)^* \sqcup\!\sqcup (\beta y)^* = (\alpha x + \beta y)^*$$

9. The theory developed here allows to pursue, for the Harmonic sums, this investigation such as

$$(\alpha y_i)^* \,\uplus\, (\beta y_j)^* = (\alpha y_i + \beta y_j + \alpha\beta y_{i+j})^*$$

THANK YOU FOR YOUR ATTENTION !

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